Definitions and units

Symbol	Meaning (first term is the most common)	SI Unit of Measure
$\mathbf{E}$	electric field	volt per meter or, equivalently, newton per coulomb
$\mathbf{B}$	magnetic field also called the magnetic induction also called the magnetic field density also called the magnetic flux density	tesla, or equivalently, weber per square meter volt•second per square meter
$\mathbf{D}$	electric displacement field also called the electric flux density	coulombs per square meter or, equivalently, newton per volt-meter
$\mathbf{H}$	magnetizing field also called auxiliary magnetic field also called magnetic field intensity also called magnetic field	ampere per meter
$\nabla\cdot$	the divergence operator	per meter (factor contributed by applying either operator)
$\nabla\times$	the curl operator	
$\frac{\partial}{\partial t}$	partial derivative with respect to time	per second (factor contributed by applying the operator)
$\mathrm{d}\mathbf{A}$	differential vector element of surface area *A*, with infinitesimally small magnitude and direction normal to surface *S*	square meters
$\mathrm{d}\mathbf{l}$	differential vector element of *path length* tangential to the path/curve	meters
ε_0	permittivity of free space, officially the electric constant, a universal constant	farads per meter
μ_0	permeability of free space, officially the magnetic constant, a universal constant	henries per meter, or Newtons per ampere squared
ρ_f	free charge density (not including bound charge)	coulombs per cubic meter
ρ	total charge density (including both free and bound charge)	coulombs per cubic meter
$\mathbf{J}_f$	free current density (not including bound current)	amperes per square meter
$\mathbf{J}$	total current density (including both free and bound current)	amperes per square meter

$Q_f(V)$	net unbalanced free electric charge in the interior of an arbitrary closed surface $S = \partial V$ (not including bound charge)	coulombs
$Q(V)$	net unbalanced electric charge in the interior of an arbitrary closed surface $S = \partial V$ (including both free and bound charge)	coulombs
$\oint_{\partial S} \mathbf{E} \cdot \mathrm{d}\mathbf{l}$	line integral of the electric field along the boundary ∂S (therefore necessarily a closed curve) of the surface S	joules per coulomb
$\oint_{\partial S} \mathbf{B} \cdot \mathrm{d}\mathbf{l}$	line integral of the magnetic field over the closed boundary ∂S of the surface S	tesla-meters
$\oint\!\!\oint_{\partial V} \mathbf{E} \cdot \mathrm{d}\mathbf{A}$	the flux of the electric field through any closed surface $S = \partial V$	joule-meter per coulomb
$\oint\!\!\oint_{\partial V} \mathbf{B} \cdot \mathrm{d}\mathbf{A}$	the flux of the magnetic field through any closed surface $S = \partial V$	tesla meters-squared or webers
$\iint_S \mathbf{B} \cdot \mathrm{d}\mathbf{A} = \Phi_{B,S}$	magnetic flux through any surface S (not necessarily closed)	webers or equivalently, volt-seconds
$\iint_S \mathbf{E} \cdot \mathrm{d}\mathbf{A} = \Phi_{E,S}$	electric flux through any surface S, not necessarily closed	joule-meters per coulomb
$\iint_S \mathbf{D} \cdot \mathrm{d}\mathbf{A} = \Phi_{D,S}$	flux of electric displacement field through any surface S, not necessarily closed	coulombs
$\iint_S \mathbf{J}_f \cdot \mathrm{d}\mathbf{A} = I_{f,s}$	net free electrical current passing through the surface S (not including bound current)	amperes
$\iint_S \mathbf{J} \cdot \mathrm{d}\mathbf{A} = I_S$	net electrical current passing through the surface S (including both free and bound current)	amperes

The Maxwell's Equations

Formulation in terms of *free* charge and current

Name	Differential form	Integral form
Gauss's law	$\nabla \cdot \mathbf{D} = \rho_f$	$\oint\!\!\oint_{\partial V} \mathbf{D} \cdot \mathrm{d}\mathbf{A} = Q_f(V)$
Gauss's law for magnetism	$\nabla \cdot \mathbf{B} = 0$	$\oint\!\!\oint_{\partial V} \mathbf{B} \cdot \mathrm{d}\mathbf{A} = 0$
Maxwell–Faraday equation (Faraday's law of induction)	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_{\partial S} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{\partial \Phi_{B,S}}{\partial t}$
Ampère's circuital law (with Maxwell's correction)	$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_{\partial S} \mathbf{H} \cdot \mathrm{d}\mathbf{l} = I_{f,S} + \frac{\partial \Phi_{D,S}}{\partial t}$

Formulation in terms of *total* charge [3]

Name	Differential form	Integral form
Gauss's law	$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$	$\oiint_{\partial V} \mathbf{E} \cdot \mathrm{d}\mathbf{A} = \frac{Q(V)}{\varepsilon_0}$
Gauss's law for magnetism	$\nabla \cdot \mathbf{B} = 0$	$\oiint_{\partial V} \mathbf{B} \cdot \mathrm{d}\mathbf{A} = 0$
Maxwell–Faraday equation (Faraday's law of induction)	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_{\partial S} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{\partial \Phi_{B,S}}{\partial t}$
Ampère's circuital law (with Maxwell's correction)	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$	$\oint_{\partial S} \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I_S + \mu_0 \varepsilon_0 \frac{\partial \Phi_{E,S}}{\partial t}$

in cgs units

Gaussian units, the equations take the following form:[47]

$$\nabla \cdot \mathbf{D} = 4\pi\rho_f$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{J}_f$$

where c is the speed of light in a vacuum. For the electromagnetic field in a vacuum, the equations become:

$$\nabla \cdot \mathbf{E} = 0$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}.$$

In this system of units the relation between displacement field, electric field and polarization density is:

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}.$$

And likewise the relation between magnetic induction, magnetic field and total magnetization is:

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}.$$

In the linear approximation, the electric susceptibility and magnetic susceptibility can be defined so that:

$$\mathbf{P} = \chi_e\mathbf{E}, \quad \mathbf{M} = \chi_m\mathbf{H}.$$

(Note that although the susceptibilities are dimensionless numbers in both cgs and SI, they have different values in the two unit systems, by a factor of 4π.) The permittivity and permeability are:

$$\epsilon = 1 + 4\pi\chi_e, \quad \mu = 1 + 4\pi\chi_m$$

so that

$$\mathbf{D} = \epsilon\mathbf{E}, \quad \mathbf{B} = \mu\mathbf{H}.$$

In vacuum, one has the simple relations $\epsilon = \mu = 1$, D=E, and B=H.

The force exerted upon a charged particle by the electric field and magnetic field is given by the Lorentz force equation:

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right),$$

where q is the charge on the particle and $\mathbf{v}$ is the particle velocity. This is slightly different from the SI-unit expression above. For example, here the magnetic field $\mathbf{B}$ has the same units as the electric field $\mathbf{E}$.

Some equations in the article are given in Gaussian units but not SI or vice-versa. Fortunately, there are general rules to convert from one to the other; see the article Gaussian units for details.

original Maxwell's equations

Maxwell's *A Dynamical Theory of the Electromagnetic Field* (1864)

[edit]

Main article: A Dynamical Theory of the Electromagnetic Field

In 1864 Maxwell published **A Dynamical Theory of the Electromagnetic Field** in which he showed that light was an electromagnetic phenomenon. Confusion over the term "Maxwell's equations" is exacerbated because it is also sometimes used for a set of eight equations that appeared in Part III of Maxwell's 1864 paper A Dynamical Theory of the Electromagnetic Field, entitled "General Equations of the Electromagnetic Field",[11] a confusion compounded by the writing of six of those eight equations as three separate equations (one for each of the Cartesian axes), resulting in twenty equations in twenty unknowns. (As noted above, this terminology is not common: Modern references to the term "Maxwell's equations" refer to the Heaviside restatements.)

The eight original Maxwell's equations can be written in modern vector notation as follows:

(A) The law of total currents

$$\mathbf{J}_{tot} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

(B) The equation of magnetic force

$$\mu \mathbf{H} = \nabla \times \mathbf{A}$$

(C) Ampère's circuital law

$$\nabla \times \mathbf{H} = \mathbf{J}_{tot}$$

(D) Electromotive force created by convection, induction, and by static electricity. (This is in effect the Lorentz force)

$$\mathbf{E} = \mu \mathbf{v} \times \mathbf{H} - \frac{\partial \mathbf{A}}{\partial t} - \nabla \phi$$

(E) The electric elasticity equation

$$\mathbf{E} = \frac{1}{\epsilon}\mathbf{D}$$

(F) Ohm's law

$$\mathbf{E} = \frac{1}{\sigma}\mathbf{J}$$

(G) Gauss's law

$$\nabla \cdot \mathbf{D} = \rho$$

(H) Equation of continuity

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

Notation

- $\mathbf{H}$ is the magnetizing field, which Maxwell called the "magnetic intensity".
- $\mathbf{J}$ is the electric current density (with $\mathbf{J}_{tot}$ being the total current including displacement current).[12]
- $\mathbf{D}$ is the displacement field (called the "electric displacement" by Maxwell).
- ρ is the free charge density (called the "quantity of free electricity" by Maxwell).
- $\mathbf{A}$ is the magnetic vector potential (called the "angular impulse" by Maxwell).
- $\mathbf{E}$ is called the "electromotive force" by Maxwell. The term electromotive force is nowadays used for voltage, but it is clear from the context that Maxwell's meaning corresponded more to the modern term electric field.
- Φ is the electric potential (which Maxwell also called "electric potential").
- σ is the electrical conductivity (Maxwell called the inverse of conductivity the "specific resistance", what is now called the resistivity).

It is interesting to note the $\mu \mathbf{v} \times \mathbf{H}$ term that appears in equation D. Equation D is therefore effectively the Lorentz force, similarly to equation (77) of his 1861 paper (see above).

When Maxwell derives the electromagnetic wave equation in his 1865 paper, he uses equation D to cater for electromagnetic induction rather than Faraday's law of induction which is used in modern textbooks. (Faraday's law itself does not appear among his equations.) However, Maxwell drops the $\mu \mathbf{v} \times \mathbf{H}$ term from equation D when he is deriving the electromagnetic wave equation, as he considers the situation only from the rest frame.

magnetic monopoles

Name	Without magnetic monopoles	With magnetic monopoles (hypothetical)
Gauss's law:	$\vec{\nabla} \cdot \vec{E} = 4\pi\rho_e$	$\vec{\nabla} \cdot \vec{E} = 4\pi\rho_e$
Gauss's law for magnetism:	$\vec{\nabla} \cdot \vec{B} = 0$	$\vec{\nabla} \cdot \vec{B} = 4\pi\rho_m$
Maxwell–Faraday equation (Faraday's law of induction):	$-\vec{\nabla} \times \vec{E} = \frac{\partial \vec{B}}{\partial t}$	$-\vec{\nabla} \times \vec{E} = \frac{\partial \vec{B}}{\partial t} + 4\pi\vec{j}_m$
Ampère's law (with Maxwell's extension):	$\vec{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{\partial t} + 4\pi\vec{j}_e$	$\vec{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{\partial t} + 4\pi\vec{j}_e$
Note: the Bivector notation embodies the sign swap, and these four equations can be written as only one equation.		

magnetic vector potential

Magnetic vector potential

[edi

The magnetic vector potential $\mathbf{A}$ is a three-dimensional vector field whose curl is the magnetic field, i.e.:

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

Since the magnetic field is divergence-free (i.e. $\nabla \cdot \mathbf{B} = 0$, called Gauss's law for magnetism), this guarantees that $\mathbf{A}$ always exists (by Helmholtz's theorem).

Unlike the magnetic field, the electric field is derived from both the scalar and vector potentials:

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}.$$

Starting with the above definitions:

$$\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0$$
$$\nabla \times \mathbf{E} = \nabla \times \left(-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) = -\frac{\partial \mathbf{B}}{\partial t}.$$

Note that the divergence of a curl will always give zero. Conveniently, this solves the second and third of Maxwell's equations automatically, which is to say that a continuous magnetic vector potential field is guaranteed not to result in magnetic monopoles.

The vector potential $\mathbf{A}$ is used when studying the Lagrangian in classical mechanics and in quantum mechanics (see Schrödinger equation for charged particles, Dirac equation, Aharonov-Bohm effect).

In the SI system, the units of $\mathbf{A}$ are volt seconds per metre ($V \cdot s \cdot m^{-1}$).

Gauge choices

[edi

Main article: Gauge fixing

It should be noted that the above definition does **not** define the magnetic vector potential uniquely because, by definition, we can arbitrarily add curl-free components to the magnetic potential without changing the observed magnetic field. Thus, there is a degree of freedom available when choosing $\mathbf{A}$. This condition is known as gauge invariance.

Magnetic scalar potential

[edi

The **magnetic scalar potential** is another useful tool in describing the magnetic field around a current source. It is only defined in regions of space in the absence of currents.

The magnetic scalar potential is defined by the equation:

$$\mathbf{B} = -\mu_0 \nabla\psi.$$

Applying Ampère's law to the above definition we get:

$$\mathbf{J} = \frac{1}{\mu_0}\nabla \times \mathbf{B} = -\nabla \times \nabla\psi = 0.$$

Solenoidality of the magnetic field leads to Laplace's equation for potential:

$$\triangle\psi = 0.$$

Since in any continuous field, the curl of a gradient is zero, this would suggest that magnetic scalar potential fields cannot support any sources. In fact, sources can be supported by applying discontinuities to the potential field (thus the same point can have two values for points along the discontinuity). These discontinuities are also known as "cuts". When solving magnetostatics problems using magnetic scalar potential, the source currents must be applied at the discontinuity.

magnetic field momentum

Field momentum

Moving charge experiences "friction" off the magnetic field. Although B exerts only perpendicular Lorentz force. This field momentum is obrtained by integrating the Poynting vector,

$$p_{field} = \frac{1}{4\pi c}\int dV E \times B$$

in non-relativistic case we can think of B only due to external sources and E comes from the charge. With electrostatic potential φ for charge q at a position r'

$$\begin{aligned} E &= -\nabla\varphi \\ \nabla^2\varphi &= -4\pi q\delta\left(r-r'\right) \end{aligned}$$

we have

$$p_{field} = \frac{1}{4\pi c}\int dV \nabla\varphi \times \mathrm{curl}\, A$$

choosing the gauge

$$\mathrm{div}\, A = 0$$

we have

$$p_{field} = \frac{q}{c}A$$

that's the physical meaning of the total momentum of a particle in a magnetic field

$$p = mv + \frac{q}{c}A$$

of course, the Hamiltonian contains the kinetic energy

$$E_{kin} = \frac{(mv)^2}{2m} = \frac{1}{2m}\left(p - \frac{q}{c}A\right)^2$$

that's why it is usually written that

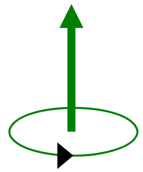
$$p \to p - \frac{q}{c}A$$

or in operational representation

$$\hat{H} = \frac{1}{2m}\left(-i\hbar\nabla - (e/c)\,\mathbf{A}\right)^2 = \frac{ie\hbar}{2mc}\left(\nabla A + A\nabla\right) + \frac{e^2}{2mc^2}A^2$$

Magnetic moment

Magnetic moment of a closed loop carrying current I: $\mathbf{M}_i = \frac{I}{2c}\oint_C \mathbf{r}\times d\mathbf{l} = IS\mathbf{n}$



Magnetic field on the axis of a loop of radius R at a distance z is: $H_z = \frac{2M_i}{\left(R^2+z^2\right)^{3/2}}$

Total magnetic moment: $\mathbf{M} = \sum \mathbf{M}_i$ (superposition principle)

origin of magnetism (except for currents…)

$$\mathbf{M}_{ion} = \gamma\hbar\mathbf{J} = -g\mu_B\mathbf{J}$$

$$\hbar\mathbf{J} = \hbar\mathbf{L} + \hbar\mathbf{S}$$

total angular momentum ($\hbar\mathbf{J}$), orbital ($\hbar\mathbf{L}$), spin ($\hbar\mathbf{S}$)

γ - gyromagnetic ratio

g – Landé factor

$$\mu_B = \frac{e\hbar}{2mc} \approx 9.27410\times10^{-21}\ \frac{erg}{G}$$ — Bohr magneton

$$g = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}$$

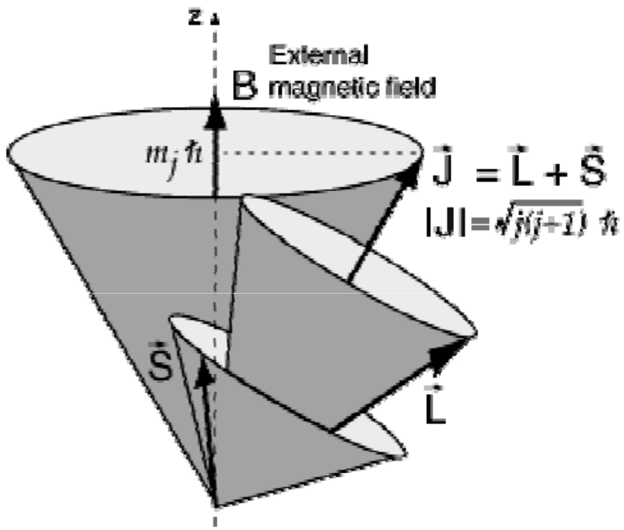
free electron:

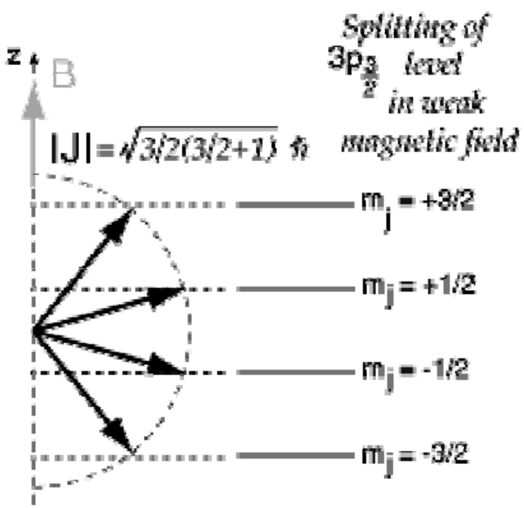
$$g = 2.0023 \approx 2.00$$

Magnetic moment:

$$M_e \approx \mu_B$$

(J=S=1/2)

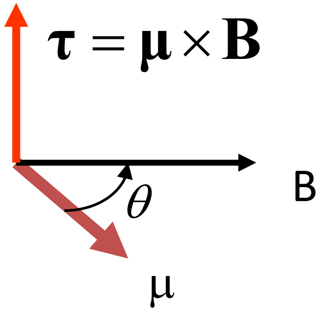




magnetic moment in a magnetic field

torque:

$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}$$



Energy: $W = -\boldsymbol{\mu}\mathbf{B} = -\mu B \cos(\theta)$

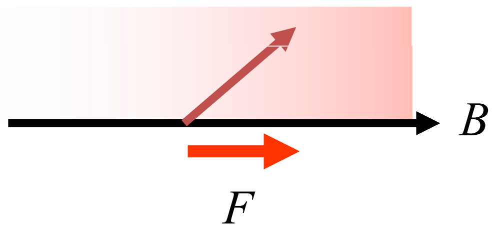
Force: $F = -grad(W) = grad(\boldsymbol{\mu}\mathbf{B})$

for example, for $\mathbf{B} = [B_x(x), 0, 0], \boldsymbol{\mu} = (\mu_x, \mu_y, 0)$

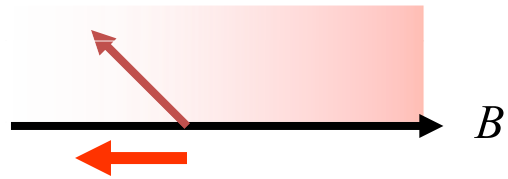
$\boldsymbol{\mu}\mathbf{B} = \mu_x B_x$ and $F = \mu_x \dfrac{dB_x}{dx}$

in inhomogeneous magnetic field

$\mu_x > 0$



$\mu_x < 0$



F changes sign

however torque aligns along the field